MATH2020A Advanced Calculus II $\frac{30}{12}$

Solution for the final exam: (For reference only)

Q1: Point out TRUE or FALSE without any proof for each statement.

a) If a function $f(x,y)$ has a critical point at $(0,0)$, then $\operatorname{div}(\nabla f)(0,0)$ is zero.

Ans: FALSE.

Counter example: $f(x,y) = x^2+y^2$

$f_x = 2x$, $f_y = 2y$, $f_x(0,0) = 0 = f_y(0,0)$, but

$\operatorname{div}(\nabla f) = \operatorname{div}(2x, 2y) = 4 \neq 0$.

b) The scalar function $f = \operatorname{div}(\vec{F})$ has the property that ∇f is perpendicular to $\vec{F}$ everywhere.

Ans: FALSE.

Counter example: $F(x,y) = (x^2, y^2)$

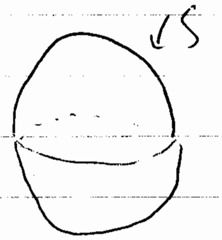
$f = \operatorname{div}\vec{F} = 2x+2y$

$\nabla f = (2,2)$, $\nabla f \not\perp \vec{F}$ in general

c) There exists a vector field $\vec{F}(x,y,z)$ defined on the three dimensional space such that every line integral $\oint_C \vec{F}\cdot d\vec{r}$ over a closed curve C is equal to 0, but not every surface integral $\iint_S \vec{F}\cdot d\vec{S}$ over a closed surface S is equal to 0.

Ans: TRUE.

Example:



Consider the vector field $\vec{F}(x,y,z) = (0,0,z)$ on the unit sphere S.

Then $\iint_S \vec{F}\cdot d\vec{S}$

$= \iint_S (0,0,z)\cdot(x,y,z)\,dA$

> 0

As $\nabla\times F \equiv 0$, $\oint_C \vec{F}\cdot d\vec{r} = 0 \ \forall$ closed curve C.

d) If $\vec{F} = \text{curl}(\vec{G})$, where $\vec{G} = (e^{e^{yz}}, 5^x z^5, \sin y \cos x)$, then $\text{div}\,\vec{F}(x,y,z) > 0$ for all (x,y,z).

Ans: FALSE.

Actually, $\text{div}(\text{curl}\,\vec{G}) = 0$ for any differentiable vector field (at least C^2)

Write $\vec{G} = (P, Q, R)$.

$$\text{curl}\,\vec{G} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

$$\text{div}(\text{curl}\,\vec{G}) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0 \quad \text{as } G \text{ at least } C^2.$$

e) The triple integral $\iiint_D (x-y)(x-z)\,dV = 0$, where D is the unit ball $\{x^2+y^2+z^2 \le 1\}$.

Ans: FALSE.

Suppose it is true.

i.e. $$\iiint_D (x-y)(x-z)\,dV = 0 \quad \text{——— (1)}$$

Then by symmetry in x, y and z, we have

$$\iiint_D (y-x)(y-z)\,dV = 0 \quad \text{— (2)}$$

(1) + (2), we have $\iiint_D (x-y)^2\,dV = 0$, which is impossible.

Hence $$\iiint_D (x-y)(x-z)\,dV \neq 0$$

f) For any vector field $\vec{F}, \vec{G}$, the following identity holds:

$$\nabla \times (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \times \vec{G} + \vec{F} \times (\nabla \times \vec{G}).$$

Ans: FALSE.

Counter example: $\vec{F} = (0, y, 0)$, $\vec{G} = (1, 0, 0)$.

Note that $\nabla \times F = 0 = \nabla \times \vec{G}$

Hence R.H.S. $\equiv 0$.

But $\vec{F} \times \vec{G} = (0, 0, -y)$ and

$\nabla \times (\vec{F} \times \vec{G}) = (-1, 0, 0) \neq 0$.

g) The flux of the vector field $\vec{F} = \nabla f$ through a closed surface $f(x,y,z)=0$ is either positive or zero if the surface is oriented so that the unit normal vector $\hat{n}$ points in the direction along which f increases.

<u>Ans:</u> TRUE.

f increases along the direction of $\hat{n}$ means

$$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n} \geq 0$$

Hence the flux of $\vec{F} = \nabla f$ given by

$\iint_S (\nabla f \cdot n)\, dA$ is either positive or zero.

h) There exists a nonzero vector field $\vec{F}$ such that $\vec{F} = \mathrm{curl}(\mathrm{curl}(\vec{F}))$

<u>Ans:</u> TRUE.

Take $\vec{F}(x,y,z) = (\sin y, 0, 0)$,

then $\vec{F} = \mathrm{curl}(\mathrm{curl}\vec{F})$ by direct calculation.

i) The line integral of the vector field $\vec{F} = (x, y, \frac{1}{z-2})$ along a circle in the xy-plane is zero.

<u>Ans:</u> TRUE.

Since $\nabla \times \vec{F} = 0$ on a simply connected neighborhood about the circle in the xy-plane, $\oint_C \vec{F} \cdot d\vec{r} = 0$

j) There exists a vector field $\vec{F}(x,y,z)$ in space such that

$$\mathrm{curl}\vec{F} = (5x + e^{y+z}, -11y, 6z + x\sin y)$$

<u>Ans:</u> TRUE.

Let $\vec{F} = (P, Q, R)$.

The question is to solve the differential equations

$$\begin{cases} R_y - Q_z = 5x + e^{y+z} & \text{——— (1)} \\ P_z - R_x = -11y & \text{——— (2)} \\ Q_x - P_y = 6z + x\sin y & \text{——— (3)} \end{cases}$$

From (1), take $R_y = 5x$, $Q_z = -e^{y+z}$. (Just a guess)

$\Rightarrow$ $R = 5xy + C^1(x,z)$, $Q = -e^{y+z} + C^2(x,y)$

From (2), $P_z - R_x = -11y$.

Assume $C^1(x,z)$ is independent on x.

$\Rightarrow$ $P_z = -11y + R_x = -6y$

$\Rightarrow$ $P = -6yz + C^3(x,y)$

From (3), $Q_x - P_y = 6z + x\sin y$

As $-P_y = 6z - C^3_y(x,y)$, assume $C^3(x,y)$ is independent on y,

we have $Q_x = x\sin y$

$\Rightarrow$ $C^2_x(x,y) = x\sin y$

$\Rightarrow$ $C^2(x,y) = \frac{1}{2}x^2\sin y + \tilde{C}^2(y)$.

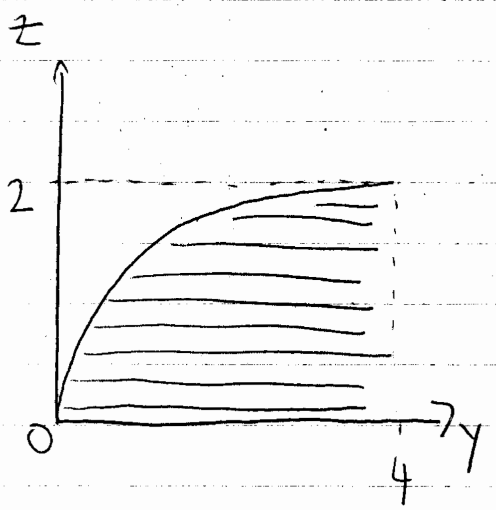
As a result, take $P = -6yz$, $Q = -e^{y+z} + \frac{1}{2}x^2\sin y$, $R = 5xy$,

we have $\text{curl}(\vec{F}) = (5x + e^{y+z}, -11y, 6z + x\sin y)$.

Remark: The solution is not unique.

Q2: Evaluate the triple integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\,dy\,dx\,dz$$



Ans:
$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\,dy\,dx\,dz$$

$$= \int_0^2 \int_{z^2}^4 \left[\frac{x^2}{2}\right]_1^3 z\cos(y^2)\,dy\,dz$$

$$= \int_0^2 \int_{z^2}^4 z\cos(y^2)\,dy\,dz$$

$$= \int_0^4 \int_0^{\sqrt{y}} z\cos(y^2)\,dz\,dy$$

$$= \int_0^4 \left[z^2\right]_0^{\sqrt{y}} \cos(y^2)\,dy$$

$$= \int_0^4 y\cos(y^2)\,dy$$

$$= \int_0^4 \cos(y^2)\, dy^2$$

$$= \sin(16) \quad //$$

Q3: Find the volume of the solid region on the quadrant $x \geq 0$ and $y \leq 0$ and bounded by the cylinder $x^2+y^2=1$, the planes $y-z=0$ and $y+z=0$

Ans: The required volume

$$= \int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_y^{-y} dz\, dy\, dx$$

$$= \int_0^1 \int_{-\sqrt{1-x^2}}^0 (-2y)\, dy\, dx$$

$$= \int_0^1 (1-x^2)\, dx$$

$$= \left[x - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} \quad //$$

Q4: Compute the surface area of the surface parametrized by

$$\vec{r}(u,v) = (3u+2v,\ 4u+v,\ \tfrac{2}{7} v^{\frac{7}{2}})$$

where $0 \leq u \leq 1$ and $u^{\frac{1}{4}} \leq v \leq 1$.

Ans: Note that $\vec{r_u} = (3, 4, 0)$,

$$\vec{r_v} = (2, 1, v^{\frac{5}{2}})$$

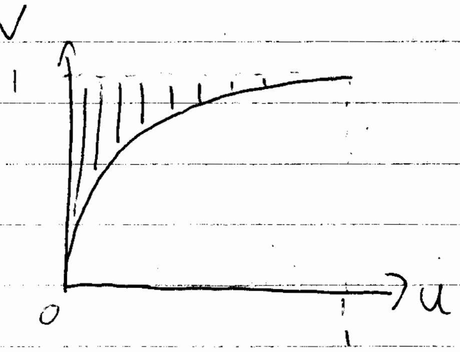
Hence $\vec{r_u} \times \vec{r_v}$

$$= \begin{vmatrix} i & j & k \\ 3 & 4 & 0 \\ 2 & 1 & v^{\frac{5}{2}} \end{vmatrix}$$

$$= (4v^{\frac{5}{2}}, -3v^{\frac{5}{2}}, -5)$$

$\|\vec{r_u} \times \vec{r_v}\| = 5\sqrt{v^5+1}$

Hence the surface area of S



$= \int_0^1 \int_{u^{\frac{1}{4}}}^1 5\sqrt{v^5+1}\, dv\, du$

$= \int_0^1 \int_0^{v^4} (5\sqrt{v^5+1})\, du\, dv$

$= \int_0^1 (5v^4\sqrt{v^5+1})\, dv$

$= \int_0^1 \sqrt{v^5+1}\, d(v^5)$

$= \left[\frac{2}{3}(v^5+1)^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}(2\sqrt{2}-1)$

//

Q5: Let the vector field $\vec{F} = (ay^2, 2y(x+z), by^2+z^2)$

a) For which values of a and b is the vector field $\vec{F}$ conservative?

b) Find a function f such that $\vec{F} = \nabla f$, for these values

c) Find the equation of a surface S with the property that for every curve C lying on S,

$$\int_C \vec{F} \cdot d\vec{r} = 0 \quad \text{for these values}$$

Ans: a) $\vec{F}$ is conservative iff $\nabla \times \vec{F} = 0$

As $\nabla \times \vec{F} = (2by - 2y, 0, 2y - 2ay)$, $\nabla \times \vec{F} = 0$ iff

$a = b = 1$

b) For $a=b=1$, $\vec{F} = (y^2, 2y(x+z), y^2+z^2)$

Let f be a potential function of $\vec{F}$

Then $\begin{cases} f_x = y^2 & \text{--- (1)} \\ f_y = 2y(x+z) & \text{--- (2)} \\ f_z = y^2+z^2 & \text{--- (3)} \end{cases}$

From (1), $f = xy^2 + C(y,z)$.

$f_y = 2xy + C_y$

By (2), $C_y = 2yz$

$\Rightarrow C = y^2 z + \tilde{C}(z)$.

$\Rightarrow f = x^2 y + y^2 z + \hat{c}(z)$.

$\Rightarrow f_z = y^2 + \tilde{C}'$.

By (3), $\tilde{C}' = z^2$

$\Rightarrow \tilde{c} = z^3/3$.

Hence $f(x,y,z) = xy^2 + y^2 z + \frac{1}{3}z^3$ is a potential function for $\vec{F}$.

c) The surface S given by $f(x,y,z) =$ constant satisfies the required property.

__Remark__: It bases on a fact that you may learn in Advanced Calculus I, that is, ∇f is perpendicular to the level set of f. Here is a proof of the fact:

Let γ be a curve on S (not necessarily closed).

Then we have $f(\gamma(t)) =$ constant

$$\Rightarrow \frac{d}{dt} f(\gamma(t)) = 0$$

$$\Rightarrow \nabla f \cdot \gamma'(t) = 0$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{\gamma} = 0.$$

__Q6__: Let S be the part of the spherical surface $x^2+y^2+z^2=4$, lying in $x^2+y^2 > 1$, which is to say outside the cylinder of radius one with axis the z-axis.

a) Compute the flux outward through S of the vector field $\vec{F} = (-y, x, z)$.

b) Compute the volume of the region between S and the cylinder.

<u>Ans</u>: a) The outward flux

$$= \iint_S \vec{F} \cdot n \, dA$$

$$= \iint_S (-y, x, z) \cdot \frac{(x, y, z)}{2} \, dA$$

$$= \frac{1}{2} \iint_S z^2 \, dA$$

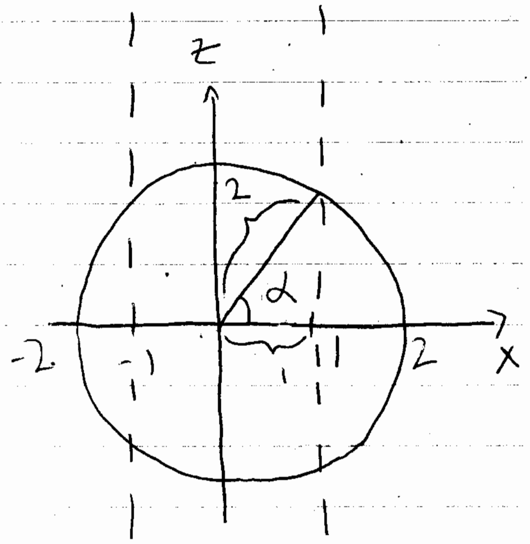
$$= \frac{1}{2} \int_0^{2\pi} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\sin^2\phi)(4\cos\phi) \, d\phi \, d\theta$$

$$= 16\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2\phi \cos\phi \, d\phi$$

$$= 16\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2\phi \, d\sin\phi$$

$$= 16\pi \left[\frac{\sin^3\phi}{3} \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 4\sqrt{3}\pi$$,,



$\cos\alpha = \frac{1}{2}$, $\alpha = \frac{\pi}{3}$.

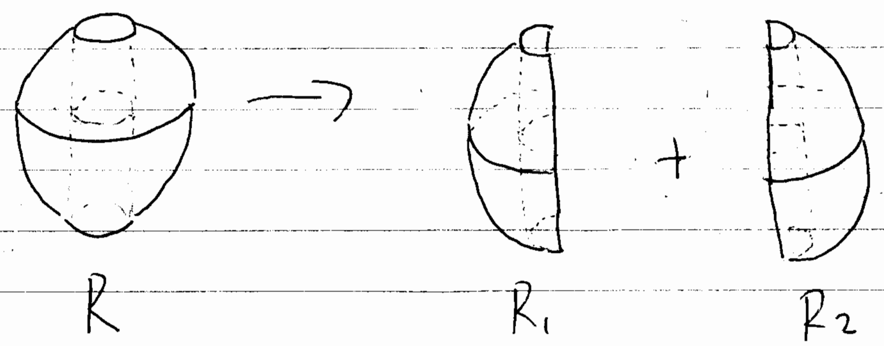
Spherical coordinates:

$\Psi(\theta, \phi)$

$= (2\cos\theta\cos\phi, 2\sin\theta\cos\phi, 2\sin\phi)$,

$\theta \in [0, 2\pi]$, $\phi \in [-\frac{\pi}{3}, \frac{\pi}{3}]$

b) Seperate the solid into two Identical parts:



As R_1 and R_2 are simply connected, divergence thm applies

$$\iiint_R \nabla\cdot F\,dV = \iiint_{R_1} \nabla\cdot F\,dV + \iiint_{R_2} \nabla\cdot F\,dV$$

$$= \iint_{\partial R_1} F\cdot n\,dA + \iint_{\partial R_2} F\cdot n\,dA$$

$$= \iint_S F\cdot n\,dA + \iint F\cdot n\,dA \quad -(*)$$

Note that $\iiint_R \nabla\cdot F\,dV = \iiint_R dV = \text{volume of } R$,

$\iint_S F\cdot n\,dA = 4\sqrt{3}\pi$ (by a)),

$\iint F\cdot n\,dA = \iint (-y,x,z)\cdot(-x,-y,0)\,dA = 0$.

Hence by (*), the required volume $= 4\sqrt{3}\pi$

Q7: Find the line integral of $\vec{F}(x,y,z) = (-y^3+\cos x, x^3, -z^3)$ along the curve $\vec{r}(t) = (\cos t, \sin t, 1-\cos t-\sin t)$ with $0 \le t \le 2\pi$.

Ans: Note that
$$\vec{F}(x,y,z) = (-y^3+\cos x, x^3, -z^3)$$
$$= (-y^3, x^3, 0) + (\cos x, 0, -z^3)$$
$$:= \vec{F}_1(x,y,z) + \vec{F}_2(x,y,z).$$

Note that $\nabla\times\vec{F}_2 = 0$. Hence $\oint_C \vec{F}_2 d\vec{r} = 0$

For $\vec{F}_1$, $\oint_C \vec{F}_1\cdot d\vec{r} = \int_0^{2\pi} (-\sin^3 t, \cos^3 t, 0)\cdot(-\sin t, \cos t, 1+\sin t-\cos t)\,dt$

$$= \int_0^{2\pi} (\sin^4 t + \cos^4 t)\,dt$$

$$= 2\int_0^{2\pi} \cos^4 t \, dt$$

$$= 2\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos 2t + \cos^2 2t)\, dt$$

$$= \frac{1}{2}\left(2\pi + \int_0^{2\pi} \cos^2 2t \, dt\right).$$

$$= \frac{1}{2}\left(2\pi + \int_0^{2\pi} \frac{1+\cos 4t}{2}\, dt\right)$$

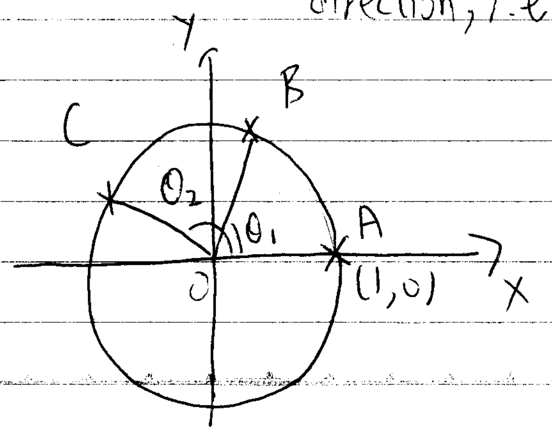
$$= \frac{1}{2}(2\pi + \pi)$$

$$= \frac{3\pi}{2}$$

Hence $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}_1 \cdot d\vec{r} + \oint_C \vec{F}_2 \cdot d\vec{r} = \frac{3\pi}{2}$,,

Q8: Find the average area of an inscribed triangle in the unit circle. Assume that each vertex of the triangle is equally likely to be at any point of the unit circle and that the location of one vertex does not affect the likelihood of another in any way.

Ans: WLOG, fix one point A at $(1,0)$, and write B and C to be $(\cos\theta_1, \sin\theta_1)$ and $(\cos\theta_2, \sin\theta_2)$ in anti-clockwise direction, i.e. $0 \le \theta_1 \le \theta_2 \le 2\pi$



Note that $\vec{AB} = (\cos\theta_1 - 1, \sin\theta_1)$,
$\vec{AC} = (\cos\theta_2 - 1, \sin\theta_2)$

So area of $\triangle ABC$

$= \frac{1}{2} \times$ area of the parallelogram formed by $\vec{AB}$ and $\vec{AC}$.

$$= \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

$$= \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ \cos\theta_1 - 1 & \sin\theta_1 & 0 \\ \cos\theta_2 - 1 & \sin\theta_2 & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2} (\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1 + \sin\theta_1 - \sin\theta_2)$$

$$= \frac{1}{2} (\sin(\theta_1 + \theta_2) + \sin\theta_1 - \sin\theta_2)$$

(As $\vec{AB} \to \vec{AC}$ is in anti-clockwise direction, by right hand grip rule, $\vec{AB} \times \vec{AC}$ has positive z-component and hence absolute sign is not needed.)

Hence the average area is given by

$$A = \frac{\int_0^{2\pi} \int_{\theta_1}^{2\pi} \frac{1}{2} (\sin(\theta_1 + \theta_2) + \sin\theta_1 - \sin\theta_2) \, d\theta_2 \, d\theta_1}{\int_0^{2\pi} \int_{\theta_1}^{2\pi} d\theta_2 \, d\theta_1}$$

Note that $\int_0^{2\pi} \int_{\theta_1}^{2\pi} \frac{1}{2} (\sin(\theta_1 + \theta_2) + \sin\theta_1 - \sin\theta_2) \, d\theta_2 \, d\theta_1$

$$= \frac{1}{2} \int_0^{2\pi} \left[-\cos(\theta_1 + \theta_2) + (\sin\theta_1)\theta_2 + \cos\theta_2 \right]_{\theta_1}^{2\pi} d\theta_1$$

$$= \frac{1}{2} \int_0^{2\pi} (-\cos\theta_1 + \cos 2\theta_1 + (2\pi - \theta_1)\sin\theta_1 + 1 - \cos\theta_1) \, d\theta_1$$

$$= \frac{1}{2} \int_0^{2\pi} (1 - \theta_1 \sin\theta_1) \, d\theta_1$$

$$= \pi + \int_0^{2\pi} \theta_1 \, d\cos\theta_1$$

$$= \pi + [\theta_1 \cos\theta_1]_0^{2\pi} - \int_0^{2\pi} \cos\theta_1 \, d\theta_1$$

$$= 3\pi$$

Hence $$A = \frac{3\pi}{\int_0^{2\pi}\int_{\theta_1}^{2\pi} d\theta_2 d\theta_1}$$

$$= \frac{3\pi}{2\pi^2}$$

$$= \frac{3}{2\pi} \quad //$$

{ This is the end of the course ☺ }